Gravitational interaction of quantum vacuum energy: How does Casimir energy fall?

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Part I: collaboration with K. V. Shajesh, P. Parashar, J. Wagner

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Introduction: Part I

The subject of Quantum Vacuum Energy (the Casimir effect) dates from the same year as the discovery of renormalized quantum electrodynamics, 1948. It puts the lie to the naive presumption that zero-point energy is not observable. On the other hand, because of the severe divergence structure of the theory, controversy has surrounded it from the beginning. Here we will deal with divergences carefully!

Meaninglessness of Self-energies?

Sharp boundaries, and even soft ones, give rise to divergences in the local energy density near the surface, which may make it impossible to extract meaningful self-energies of single objects, such as the perfectly conducting sphere considered by Boyer. [Graham, Jaffe, et al. (21st century), Deutsch and Candelas (1979)].

In fact, it now appears that these surface divergences can be dealt with successfully in a process of renormalization. See for example, PRD **88**, 025039, 045030 (2013).

Coupling to gravity

Gravity couples to the local energy-momentum tensor, and such surface divergences promise serious difficulties.

- How does the completely finite Casimir interaction energy of a pair of parallel conducting plates, as well as the divergent self-energies of non-ideal plates, couple to gravity?
- For a beginning of the renormalization of Einstein's equations resulting from singular Casimir surface energy densities see Estrada et al., J. Phys. A **41**, 164055 (2008).

Casimir Stress Tensor for $\parallel$ Plates

Brown and Maclay [a] showed that, for parallel perfectly conducting plates separated by a distance a in the z-direction, the electromagnetic stress tensor acquires the vacuum expectation value between the plates

$$\langle T^{\mu\nu}\rangle = \frac{\mathcal{E}_c}{a}\text{diag}(1,-1,-1,3), \quad \mathcal{E}_c = -\frac{\pi^2}{720a^3}\hbar c.$$

Outside the plates the value of $\langle T^{\mu\nu}\rangle = 0$.

[a] L. S. Brown and G. J. Maclay, Phys. Rev. **184**, 1272 (1969)

Variational principle

Now we turn to the question of the gravitational interaction of this Casimir apparatus. It seems this question can be most simply addressed through use of the gravitational definition of the energy-momentum tensor,

$$\delta W_m \equiv -\frac{1}{2}\int (dx)\sqrt{-g}\,\delta g^{\mu\nu} T_{\mu\nu} = \frac{1}{2}\int (dx)\sqrt{-g}\,\delta g_{\mu\nu} T^{\mu\nu}.$$

For a weak field,

$$g_{\mu\nu} = \eta_{\mu\nu} + 2h_{\mu\nu}$$

(Schwinger's definition of $h_{\mu\nu}$).

Gravitational energy

So if we think of turning on the gravitational field as the perturbation, we can ignore $\sqrt{-g}$. The gravitational energy, for a static situation, is therefore given by ($\delta W = -\int dt\, \delta E$)

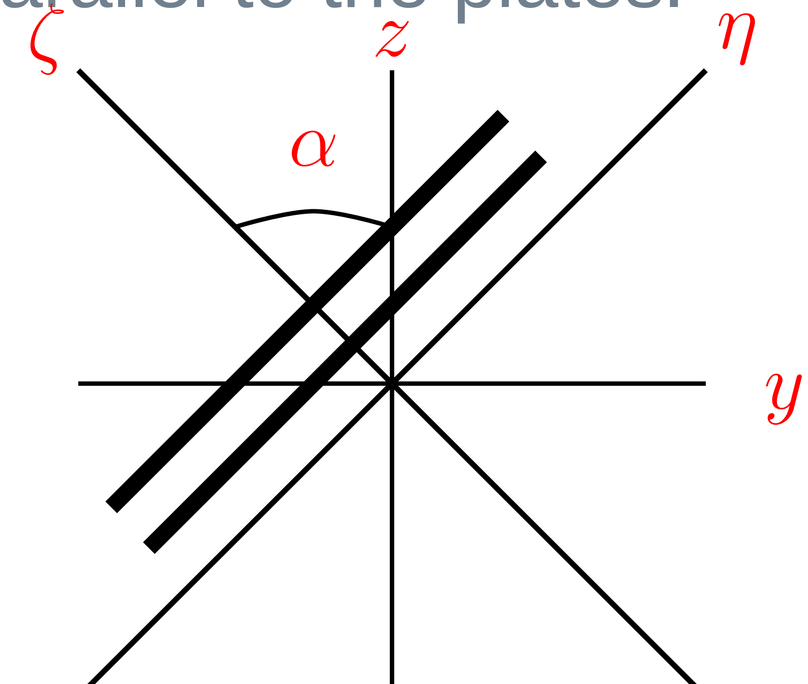
$$E_g = -\int (d\mathbf{x}) h_{\mu\nu} T^{\mu\nu}.$$

The Fermi metric describes an inertial coordinate system:

$$h_{00} = -gz, \quad h_{0i} = h_{ij} = 0.$$

Cartesian coordinate systems

The Cartesian coordinate system attached to the earth is (x, y, z), where z is the direction of $-\mathbf{g}$. The Cartesian coordinates associated with the Casimir apparatus (plate separation a, length L) are (ζ, η, χ), where ζ is normal to the plates, and η and χ are parallel to the plates.



Gravitational energy of apparatus

Now we calculate the gravitational energy

$$E_g = \int (d\mathbf{x}) g z T^{00} = g\mathcal{E}_c L^2 \zeta_0 \cos\alpha + K,$$

where K is a constant, independent of the center of the apparatus ζ_0, $z_0 = \zeta_0 \cos\alpha$. Thus, the gravitational force per area $A = L^2$ on the apparatus is independent of orientation

$$\frac{F}{A} = -\frac{\partial E_g}{A \partial z_0} = -g\mathcal{E}_c,$$

a small upward push.

Rindler Coordinates

Relativistically, uniform acceleration is described by hyperbolic motion

$$t = \xi \sinh \tau, \qquad z = \xi \cosh \tau,$$

which corresponds to the metric

$$dt^2 - dz^2 = \xi^2 d\tau^2 - d\xi^2.$$

d'Alembertian operator has cylindrical form

$$-\left(\frac{\partial}{\partial t}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2 = -\frac{1}{\xi^2}\left(\frac{\partial}{\partial \tau}\right)^2 + \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial}{\partial \xi}\right).$$

Single Accelerated δ-Plate

For a single semitransparent plate at ξ_1, the Green's function can be written as

$$G(x,x') = \int \frac{d\omega}{2\pi}\frac{d^2k}{(2\pi)^2} e^{-i\omega(\tau-\tau')} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')_\perp} g(\xi,\xi'),$$

where the reduced Green's function satisfies

$$\left[-\frac{\omega^2}{\xi^2} - \frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right) + k^2 + \lambda\delta(\xi-\xi_1)\right] g = \frac{1}{\xi}\delta(\xi-\xi'),$$

which we recognize as just the semitransparent cylinder problem with $m \to \zeta = -i\omega$ and $\kappa \to k$.

Energy-Momentum Tensor

The canonical energy-momentum for a scalar field is given by $T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\frac{1}{\sqrt{-g}}\mathcal{L}$, where the Lagrange density includes the δ-function potential. Using the equations of motion:

$$T_{00} = \frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 - \frac{1}{2}\phi\frac{\partial^2}{\partial\tau^2}\phi + \frac{\xi}{2}\frac{\partial}{\partial\xi}\left(\phi\xi\frac{\partial}{\partial\xi}\phi\right) + \frac{\xi^2}{2}\boldsymbol{\nabla}_\perp\cdot(\phi\boldsymbol{\nabla}_\perp\phi).$$

$\langle T_{\mu\nu}\rangle$ follows from $\langle\phi(x)\phi(y)\rangle = \frac{1}{i}G(x,y)$.

Force Density

The force density is given by

$$f_\lambda = -\frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}T^\nu{}_\lambda) + \frac{1}{2}T^{\mu\nu}\partial_\lambda g_{\mu\nu},$$

so the gravitational force on the system is

$$\mathcal{F} = \int d\xi\,\xi f_\xi = -\int \frac{d\xi}{\xi^2}T_{00}$$

$$= \int d\xi\,\xi \int \frac{d\hat\zeta\, d^2k}{(2\pi)^3}\hat\zeta^2 g(\xi,\xi) \quad (\zeta = \hat\zeta\xi);$$

g is given in terms of modified Bessel functions.

Flat-space Limit for 2 $\|$ Plates

In the weak acceleration limit, the Green's function reduces to exactly the expected result, for $\xi_1 < \xi, \xi' < \xi_2$ $(a = \xi_2 - \xi_1)$

$$\xi_0 g(\xi,\xi') \to \frac{1}{2\kappa} e^{-\kappa|\xi-\xi'|} + \frac{1}{2\kappa\tilde{\Delta}} \left[\frac{\lambda_1\lambda_2}{4\kappa^2} 2\cosh\kappa(\xi-\xi') \right.$$

$$- \frac{\lambda_1}{2\kappa}\left(1+\frac{\lambda_2}{2\kappa}\right) e^{-\kappa(\xi+\xi'-2\xi_2)} - \frac{\lambda_2}{2\kappa}\left(1+\frac{\lambda_1}{2\kappa}\right) e^{\kappa(\xi+\xi'-2\xi_1)}$$

$$\tilde{\Delta} = \left(1+\frac{\lambda_1}{2\kappa}\right)\left(1+\frac{\lambda_2}{2\kappa}\right) e^{2\kappa a} - \frac{\lambda_1\lambda_2}{4\kappa^2},$$

The flat space limit also holds outside the plates.

Explicit Force on 2-plate Apparatus

$$\mathcal{F} = \frac{1}{96\pi^2 a^3}\int_0^\infty dy\, y^3 \frac{1 + \frac{1}{y+\lambda_1 a} + \frac{1}{y+\lambda_2 a}}{\left(\frac{y}{\lambda_1 a}+1\right)\left(\frac{y}{\lambda_2 a}+1\right)e^y - 1}$$
$$- \frac{1}{96\pi^2 a^3}\int_0^\infty dy\, y^2 \left[\frac{1}{\frac{y}{\lambda_1 a}+1} + \frac{1}{\frac{y}{\lambda_2 a}+1}\right]$$
$$= -(\mathcal{E}_c + \mathcal{E}_{d1} + \mathcal{E}_{d2}),$$

which is just the negative of the Casimir energy of the two semitransparent plates, including divergent parts associated with each plate.

Mass Renormalization

The divergent term simply renormalizes the mass of each plate:

$$\begin{aligned} E_{\text{total}} &= m_1 + m_2 + \mathcal{E}_{d1} + \mathcal{E}_{d2} + \mathcal{E}_c \\ &= M_1 + M_2 + \mathcal{E}_c, \end{aligned}$$

and thus the gravitational force on the entire apparatus obeys the equivalence principle

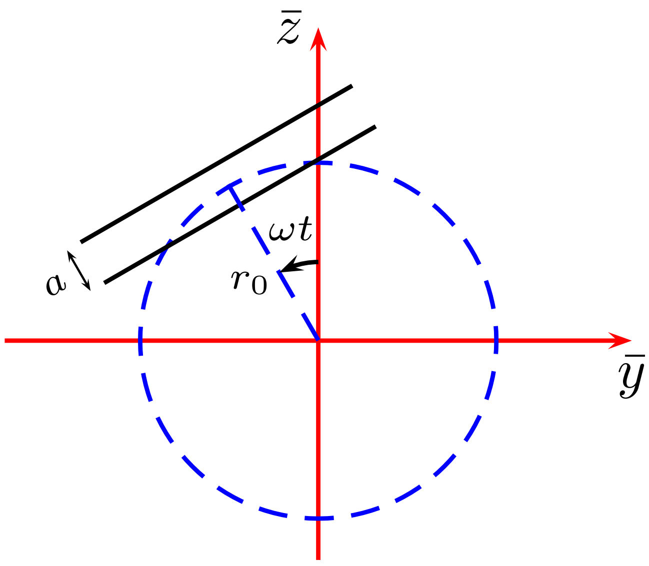
$$g\mathcal{F} = -g(M_1 + M_2 + \mathcal{E}_c).$$

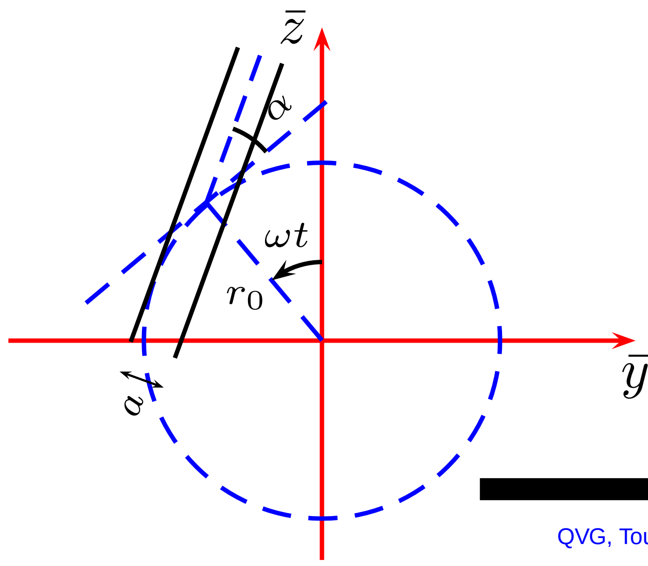
Centripetal Acceleration

Casimir apparatus undergoing centripetal acceleration ($\omega r \ll 1$)

$$\mathbf{F} = -\omega^2 \int d^3x\, \mathbf{r}\, t^{00}(\mathbf{r})$$

$$= -\omega^2 \mathbf{r}_{CM}(m_1 + m_2 + E_{d1} + E_{d2} + E_c)$$





Part II. Divergences

The above analysis may be rightly criticized for not dealing with divergences properly, and performing manipulations with divergent integrals. In the rest of this talk, I will remedy this situation. We will consider two semi-transparent plates, interacting with a massless scalar field, with the potential

$$V = -\frac{\lambda_1}{2}\delta(z)\phi^2 - \frac{\lambda_2}{2}\delta(z-a)\phi^2.$$

Green's function, $0 < z, z' < a$

The Green's function

$$G(\mathbf{r}, \mathbf{r}'; \omega) = \int \frac{(d\mathbf{k}_\perp)}{(2\pi)^2} e^{i\mathbf{k}_\perp \cdot (\mathbf{r}-\mathbf{r}')_\perp} g(z, z'),$$

$$g = \frac{1}{2\kappa} e^{-\kappa|z-z'|} + \frac{1}{2\kappa\Delta}\left[\frac{\lambda_1\lambda_2}{(2\kappa)^2} 2\cosh\kappa(z-z') \right.$$
$$\left. -\frac{\lambda_1}{2\kappa}\left(1+\frac{\lambda_2}{2\kappa}\right)e^{\kappa(2a-z-z')} - \frac{\lambda_2}{2\kappa}\left(1+\frac{\lambda_1}{2\kappa}\right)e^{\kappa(z+z')}\right]$$

,

Denominator

where

$$\Delta = \left(1 + \frac{\lambda_1}{2\kappa}\right)\left(1 + \frac{\lambda_2}{2\kappa}\right) e^{2\kappa a} - \frac{\lambda_1 \lambda_2}{(2\kappa)^2},$$

with

$$\kappa = \sqrt{k_\perp^2 + \zeta^2}, \quad \omega \to i\zeta.$$

Point-split regularization

To define the integrals, we adopt point splitting in the time and the transverse directions (but not the z direction:

$$\tau = t_E - t_E' \to 0, \quad \mathbf{R}_\perp = (\mathbf{r} - \mathbf{r}')_\perp \to 0.$$

The energy is given by the general formula (more about this later)

$$E = -\int \frac{d\zeta}{2\pi} \int (d\mathbf{r}) \zeta^2 e^{i\zeta\tau} G(\mathbf{r}, \mathbf{r}')\Big|_{\mathbf{r}' \to \mathbf{r}}.$$

Formula for energy

Here the energy/area is readily seen to be given by

$$\mathcal{E} = -\int \frac{d\zeta(d\mathbf{k}_\perp)}{(2\pi)^3}\zeta^2 e^{i\zeta\tau} e^{i\mathbf{k}_\perp\cdot\mathbf{R}_\perp}\left\{\frac{L_z}{2\kappa}\right.$$
$$+\frac{1}{4\kappa^2\Delta}\left[4(\kappa a+1)\frac{\lambda_1\lambda_2}{(2\kappa)^2}\right.$$
$$\left.\left.-2e^{2\kappa a}\left(\frac{\lambda_1+\lambda_2}{2\kappa}+2\frac{\lambda_1\lambda_2}{(2\kappa)^2}\right)\right]\right\}.$$

Regulator function

If we adopt polar coordinates in $\boldsymbol{\kappa} = (\zeta, \mathbf{k}_\perp)$ space, and similarly let $\boldsymbol{\Delta} = (\tau, \mathbf{R}_\perp)$, we encounter ($\alpha$ and β are the spherical angles for $\boldsymbol{\kappa}$.)

$$f(\gamma) = \int_{-1}^{1} d\cos\alpha \int_{0}^{2\pi} d\beta \cos^2\alpha \, e^{i\boldsymbol{\kappa}\cdot\boldsymbol{\Delta}} \rightarrow \frac{4\pi}{3},$$

where γ is the angle between $\boldsymbol{\Delta}$ and the time axis. Thus, $\gamma = 0$ corresponds to time-splitting regularization, $\gamma = \pi/2$ to tranverse space-splitting. The limit is as $|\boldsymbol{\Delta}| = \delta \rightarrow 0$.

Splitting in neutral direction

It is easy to see that

$$f(0) = \frac{d}{d\delta}\left[\delta f(\pi/2)\right],$$

so in the following we will only consider $\delta = \pi/2$, space-splitting (neutral direction). Explicitly then,

$$f(\pi/2) = 4\pi\left(-\frac{\cos\kappa\delta}{(\kappa\delta)^2} + \frac{\sin\kappa\delta}{(\kappa\delta)^3}\right).$$

Bulk divergence

First we look at the Weyl, or bulk term, that would be present with no boundaries,

$$E_W(\gamma = \pi/2) = -\frac{V}{8\pi^2}\int d\kappa^2\,\kappa^2 f(\pi/2)\frac{1}{2}\kappa = -\frac{V}{2\pi\delta^4},$$

just as expected. Then

$$E_W(\gamma = 0) = \frac{3}{2\pi^2}\frac{V}{\delta^4},$$

as is familiar.

Self and interaction energy

It is then straightforward to calculate the balance of the energy/area ($y = 2\kappa a$):

$$\mathcal{E} - \mathcal{E}_W = \frac{1}{128\pi^3 a^3}\int_0^\infty dy\, y^2 f(\gamma)\left(\frac{1}{\frac{y}{\lambda_1 a}+1} + \frac{1}{\frac{y}{\lambda_2 a}+1}\right)$$

$$-\frac{1}{96\pi^3 a^3}\int_0^\infty dy\, y^3 \frac{1+\frac{1}{y+\lambda_1 a}+\frac{1}{y+\lambda_2 a}}{\left(\frac{y}{\lambda_1 a}+1\right)\left(\frac{y}{\lambda_2 a}+1\right)e^y - 1}.$$

We have set the cutoff to zero in the second, finite term. That term is the same a given previously.

Divergent terms

The divergent term is the sum of contributions from each plate separately,

$$\mathcal{E}_S = \frac{\lambda}{16\pi^2}\left(\frac{1}{\delta^2} - \frac{\lambda}{8}\frac{\pi}{\delta} - \frac{\lambda^2}{12}\ln\delta\right),$$

for $\gamma = \pi/2$. This is for finite λ. In the Dirichlet limit, $\lambda \to \infty$, the self-energy is more divergent:

$$\mathcal{E}_S = \frac{1}{8\pi}\frac{1}{\delta^3}.$$

$$\mathcal{E} = \mathcal{E}_W + \mathcal{E}_{S1} + \mathcal{E}_{S2} + \mathcal{E}_{\text{int}}.$$

Local energy density

To answer the question of how Casimir energy interacts with gravity, we have to look at local quantities. The stress tensor, including the conformal term, is

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}g^{\mu\nu}\partial_\lambda\phi\partial^\lambda\phi - \xi(\partial^\mu\partial^\nu - g^{\mu\nu}\partial^2)\phi^2.$$

Then, the Fourier-transformed expectation value of the stress tensor,

$$\langle T^{\mu\nu}\rangle = \int \frac{d\zeta}{2\pi}\frac{(d\mathbf{k}_\perp)}{(2\pi)^2}e^{i\zeta\tau}e^{i\mathbf{k}_\perp\cdot\mathbf{R}}t^{\mu,\nu}(z,z')\Big|_{z'\to z},$$

Quantum mechanical replacement

with the quantum-mechanical replacement

$$\langle \phi(\mathbf{r})\phi(\mathbf{r}')\rangle = \frac{1}{i}G(\mathbf{r},\mathbf{r}').$$

Energy density $z < 0$

$$u(z) = t^{00}(z,z) = \frac{1}{2}(-\zeta^2 + k_\perp^2 + \partial_z\partial_{z'})g(z,z')\Big|_{z'\to z}$$
$$- \xi\partial_z^2 g(z,z).$$

For $z < 0$ ($g(\pi/2) = 4\pi\sin\kappa\delta/\kappa\delta$):

$$u(z<0) = -\frac{1}{16\pi^3}\int_0^\infty d\kappa\,\kappa^3\,\overbrace{[(1-4\xi)g(\gamma) - f(\gamma)]}^{\to -16\pi(\xi-1/6)}$$
$$\times\frac{e^{2\kappa z}}{1+\frac{\lambda_1}{2\kappa}}\left[\frac{\lambda_1}{2\kappa} + \frac{1}{\Delta}\frac{\lambda_2}{2\kappa}\right].$$

Energy density inside

Similarly, for $0 < z < a$, if we are not too close to the plates, we can drop the cutoff:

$$u = \frac{1}{4\pi^2}\int_0^\infty d\kappa \frac{\kappa^3}{\Delta}\left\{ -\frac{2}{3}\frac{\lambda_1\lambda_2}{(2\kappa)^2} + 4\left(\xi - \frac{1}{6}\right)\left[\frac{\lambda_1}{2\kappa}\left(1+\frac{\lambda_2}{2\kappa}\right)e^{2\kappa(a-z)} + \frac{\lambda_2}{2\kappa}\left(1+\frac{\lambda_1}{2\kappa}\right)e^{2\kappa z}\right]\right\}.$$

Energy density for $z > a$

Again, not too close to the boundary,

$$u = \frac{\xi - 1/6}{\pi^2} \int_0^\infty \frac{d\kappa\,\kappa^3}{1 + \lambda_2/2\kappa} \left(\frac{\lambda_2}{2\kappa} + \frac{\lambda_1}{2\kappa} \frac{1}{\Delta} \right) e^{2\kappa(a-z)}.$$

Dirichlet boundaries: $\lambda \to \infty$

$$u(z<0) = \frac{3}{8\pi^2}\frac{\xi - 1/6}{z^4},$$

$$u(0<z<a) = -\frac{\pi^2}{1440a^4} + \frac{3}{8\pi^2 a^4}(\xi - 1/6) \times[\zeta(4, z/a) + \zeta(4, 1-z/a)],$$

$$u(z>a) = \frac{3}{8\pi^2}\frac{\xi - 1/6}{(a-z)^4},$$

exhibiting the familiar quartic divergences in the energy density as the boundary is approached.

Divergences telescope away

It may be useful to see how the usual argument works for getting rid of the divergent terms (which are zero for the conformal case, $\xi = 1/6$):

$$\int_{-\infty}^{\infty} dz \left[u - \frac{E_C}{a} \theta(z)\theta(a-z) \right]$$

$$= \frac{3}{8\pi^2}(\xi - 1/6) \left\{ \int_{-\infty}^{a} \frac{dz}{z^4} + \int_0^{\infty} \frac{dz}{(a-z)^4} + \int_0^a \frac{dz}{a^4} \left[\zeta(4, 1 + z/a) + \zeta(4, 2 - z/a) \right] \right\}.$$

Hurwitz zeta function

From the definition of the Hurwitz zeta function,

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s},$$

It is easy to see the above collapses to

$$\mathcal{E}_{\rm surf} = \frac{2}{8\pi^2}(\xi - 1/6) I_\epsilon,$$

where $I_\epsilon = \int_{-\epsilon}^{\epsilon} dz/z^4$, while ill-defined, is a constant independent of the plate separation a. The following will give meaning to this.

Regulation of surface divergences

Using the regulator, we can tame the surface divergences. For example, in strong coupling,

$$u(z<0)=\frac{3}{8\pi^2}\frac{(\xi-1/6)}{(z^2+\delta^2/4)^2}+\frac{7}{32\pi^2}(1-4\xi)\frac{\delta^2}{(z^2+\delta^2/4)^3},$$

which reduces to the previous result if $|z| \gg \delta$, and is finite at $z=0$.

Surface terms

Except in the Dirichlet limit, one must include a term that resides exactly on the boundary.

$$T^{00} = \frac{1}{2}\partial^0\phi\partial^0\phi + \frac{1}{2}\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi + \frac{1}{2}V\phi^2 - \xi\nabla^2\phi^2.$$

EOM gives an additional surface term:

$$\int_V (d\mathbf{r})\langle T^{00}\rangle = \int_V (d\mathbf{r})\frac{2\omega^2}{2i}G(\mathbf{r},\mathbf{r}) + \frac{1-4\xi}{2i}\int_{\partial V} d\mathbf{S}\cdot\boldsymbol{\nabla}G(\mathbf{r},\mathbf{r}')\bigg|_{\mathbf{r}'\to\mathbf{r}}.$$

Effect of surface terms

Including the contribution from the bulk term, we get here an additional contribution to the energy that resides exactly on the surface:

$$\Delta\mathcal{E} = -\frac{1-4\xi}{2}\int_{-\infty}^{\infty}\frac{d\zeta}{2\pi}\sum\frac{\partial}{\partial z}g(z,z')\bigg|_{z'\to z}$$

$$= -\frac{1-4\xi}{32\pi^3}\int_0^{\infty}d\kappa\kappa g(\gamma)\left[-\frac{\lambda_1}{1+\lambda_1/2\kappa}-\frac{\lambda_2}{1+\lambda_2/2\kappa}\right.$$

$$\left.+\frac{1}{\Delta}\frac{\lambda_1\lambda_2}{2\kappa}\left(\frac{1}{1+\lambda_1/2\kappa}+\frac{1}{1+\lambda_2/2\kappa}\right)\right].$$

Total energy recovered

Combining these contributions is straightforward:

$$\int_{-\infty}^{\infty} dz\, u(z) + \Delta\mathcal{E} = \mathcal{E} - \mathcal{E}_W$$

$$= \frac{1}{128\pi^3 a^3} \int_0^\infty dy\, y^2 f(\gamma) \left\{ \frac{1}{1 + y/\lambda_1 a} + \frac{1}{1 + y/\lambda_2 a} - y \frac{1 + \frac{1}{y+\lambda_1 a} + \frac{1}{y+\lambda_2 a}}{(1 + y/\lambda_1 a)(1 + y/\lambda_2 a)e^y - 1} \right\}.$$

This is exactly the result obtained directly. For second term, $f(\gamma) \to 4\pi/3$.

Surface term nontrivial

It is important to recognize that the surface term does not only contribute to the “divergent” self energies, but to the finite interaction energy as well. If we naively only included the integrated local energy density and just dropped the divergent terms, we would get

$$\int_{-\infty}^{\infty} dz\, u \qquad \boxed{\text{correct for } \xi = 1/4, \text{ where ST} = 0}$$

$$= -\frac{1}{96\pi^3 a^3}\int_0^\infty dy\, y^3 \frac{1 + \frac{12(\xi - 1/6)}{y+\lambda_1 a} + \frac{12(1-1/6)}{y+\lambda_2 a}}{\left(\frac{y}{\lambda_1 a}+1\right)\left(\frac{y}{\lambda_2 a}+1\right)e^y - 1}.$$

How does surface energy fall?

- For $\xi = 1/4$ there is no surface term.
- For $\xi = 0$ surface term is included in potential contribution to stress tensor.
- For $\lambda \to \infty$ surface term vanishes.
- It is not yet clear what happens in general.

Conclusions

- We have found an extremely simple answer to how Casimir energy gravitates: just like any other form of energy,

$$\frac{F}{A} = -g\mathcal{E}_c.$$

This result is independent of the orientation of the Casimir apparatus relative to the gravitational field. This refutes the claim sometimes attributed to Feynman that virtual photons do not gravitate.

- Previous arguments were formal, in that divergent self energies were not properly defined. We have now regulated everything consistently, for both the global and local description.
- Although gravitational energies have a certain ill-defined character, being gauge- or coordinate-variant, this result is obtained for a Fermi observer, the relativistic generalization of an inertial observer.

- This conclusion is supported by an explicit calculation in Rindler coordinates, describing a uniformly accelerated observer. This demonstrates, quite generally, that the total Casimir energy, including the divergent parts, which renormalize the masses of the plates, possesses the gravitational mass demanded by the equivalence principle. The gravitational effects of the surface and cutoff-dependent terms still require investigation.
- The inertial properties of Casimir energies further are confirmed by considering centripetal acceleration.