REMARKS

on the

EQUIVALENCE PRINCIPLE

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I. NEWTONIAN GRAVITY

- Inertial mass : m^{in} determined in, say, collision experiments.

$$\rho^{in} \equiv \Sigma_a m_a^{in}/V \quad , \quad M^{in} = \int dV \rho^{in}$$

- Gravitational "charge"/mass : m. Pisa etc : $m^{in} = m$ to 10^{-12}

Imposing $m^{in} = m$ for elementary objects : WEP.

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Two questions:

(a) If $M^{in} \neq M$ for astronomical bodies, what observational consequences ?

(b) Which theories impose $M^{in} = M$ for astronomical bodies ? (SEP)

Answer depends on the theory :

$M^{in} = M$ for Newton's, not for 'extended gravity theories'.

Question (a) : **If $M^{in} \neq M$, what observational consequences?**

Test, non gravitationally bound, objects are characterized by m^{in} and m.

EOM of such "probes" : $m_a^{in}\ddot{\vec{r}} = -m_a \nabla U$. Pisa etc: $m^{in} = m$, WEP.

Is that still true for astronomical bodies ?

or : Do the Moon and the Earth fall the same way in the field of the Sun ?

Laplace's approach (1825):

Suppose it is NOT true.

Consider Newton's gravity theory as Coulomb's (where the electric charge is not equal to its inertial mass). Hence WEP violated.

What are the gravitational fields created by the Moon, the Earth and the Sun ? What are their relative motions ?

• First step ("inner problem") : *the field created by an extended object*

Poisson gravitational field equation : $\triangle U = 4\pi G\rho$ NB : $M^{in} = \int dV \rho^{in}$

Euler (static) matter field equation : $\nabla p = -\rho\nabla U$

Choose an equation of state (e.g. $\rho = const.$). Find

$$U(r) = -\frac{GM}{r} \text{ for } r \geq R \quad \text{with} \quad M \equiv \int dV \rho \neq M^{in}$$

EOM of "probes" : $m^{in}\ddot{\vec{r}} = -m\nabla U$

Hence M is the "Kepler" or "active" gravitational mass

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• Second step ("N-body pb") : *"skeletonize" N extended bodies*

$$\rho^{in} = \Sigma_a M_a^{in} \delta_3(\vec{r} - \vec{r}_a(t)),\ \rho = \Sigma_a M_a \delta_3(\vec{r} - \vec{r}_a(t))$$

and eom : $M_a^{in}\,\ddot{\vec{r}}_a = -M_a \nabla U \quad , \quad \triangle U = 4\pi G\, \Sigma_a M_a \delta_3(\vec{r} - \vec{r}_a(t))$

- Third step: Earth-Moon / Sun system

$$\ddot{\vec{r}}_{ME} = -GM^* \frac{\vec{r}_{ME}}{r^3_{ME}} + GM_S\left(\frac{\vec{r}_{ES}}{r^3_{ES}} - \frac{\vec{r}_{MS}}{r^3_{MS}}\right) + GM_S\left(\epsilon_E \frac{\vec{r}_{ES}}{r^3_{ES}} - \epsilon_M \frac{\vec{r}_{MS}}{r^3_{MS}}\right)$$

with $\quad M^* \equiv M_E(M/M^{in})|_M + M_M(M/M^{in})|_E$ and $\epsilon = M/M^{in} - 1$.

First term : EOM of Earth-Moon in absence of the Sun.

Second term : tidal effects due to finite size of the "freely falling elevator". Absent if $\vec{r}_{ES} \approx \vec{r}_{MS}$: motion of gravitationally bound Earth-Moon system does not depend on the presence of exterior "spectator" bodies which are "effaced". This is the SEP.

Third term : present even if $\vec{r}_{ES} \approx \vec{r}_{MS}$; violates SEP.

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Earth-Moon laser ranging : $\left|\frac{M_E}{M_E^{in}} - \frac{M_M}{M_M^{in}}\right| < 5.5 \times 10^{-13}$.

Question (b) : **Does the theory impose $M = M^{in}$?**

Laplace : $\rho \neq \rho^{in}$ with the Newton-Coulomb $1/r^2$ law for gravity.

Suppose now: $\rho = \rho^{in}$ but law for gravity $\neq 1/r^2$.

The example of non-linear coupling:

$$S = S_g + S_m, \;\; S_g = \int dt\, L_g, \quad L_g = \int dV \mathcal{L}_g, \quad \mathcal{L}_g = -\frac{1}{8\pi G}(\nabla U)^2$$

$$S_m = \int dt\, L_m, \quad L_m = \int dV \mathcal{L}_m, \quad \mathcal{L}_m = \frac{1}{2}\rho^{in} v^2 - \rho^{in}\, V(U)$$

(Laplace lagrangian : $\quad \mathcal{L}_m = \frac{1}{2}\rho^{in} v^2 - \rho\, U$.)

Expand $V(U) = U + \frac{a_2}{2c^2} U^2 + \cdots$

Question: in such a theory EOM of a probe is $\ddot{\vec{r}} = -\nabla U$ (WEP).
Is is still true for astronomical bodies ? Answer : NO.

• First step ("inner problem") : *the field created by an extended object*

Modified Poisson equation : $\triangle U = 4\pi G \rho^{in} \frac{dV}{dU}$ (NB : $M^{in} \equiv \int dV \rho^{in}$)

Choose an EOS. Iterate. Find (for e.g. $\rho^{in} = const.$) : $U(r) = -\frac{GM}{r}$

with $$M = M^{in}\left(1 + 2a_2 \frac{W_{grav}}{c^2 M} + \cdots\right)$$ and $W_{grav} = \frac{1}{2}\int \rho^{in} U dV.$

$M \neq M^{in}$ for astronomical bodies with gravitational self-energy.

• Second step ("N-body pb") : *"skeletonization" of N extended bodies*

$$\rho^{in} = \Sigma_a M_a^{in} \delta_3(\vec{r} - \vec{r}_a(t)), \quad \rho = \Sigma_a M_a \delta_3(\vec{r} - \vec{r}_a(t))$$

with: $M = M^{in}(1 + 2a_2\epsilon)$ where $\epsilon \equiv \frac{W_{grav}}{c^2 M}$ constant.

eom : $$\triangle U = 4\pi G\, \Sigma_a M_a \frac{dV}{dU}\, \delta_3(\vec{r} - \vec{r}_a(t)), \quad M_a^{in}\, \ddot{\vec{r}}_a = -M_a \frac{dV}{dU} \nabla U$$

• Third step "Laplace" effect (ignoring tidal corrections) :

$$\ddot{\vec{r}}_{ME} = -GM^* \left[1 - a_2 \left(\frac{2GM_S}{c^2 r_{ES}} + \frac{GM^*}{c^2 r_{EM}}\right)\right] \frac{\vec{r}_{ME}}{r^3_{ME}} +$$

$$+2a_2 \left[(\epsilon_E - \epsilon_M) + \frac{G(M_E - M_M)}{2c^2 r_{EM}}\right] GM_S \frac{\vec{r}_{ES}}{r^3_{ES}}$$

Extra-terms are negligible. Hence the "standard" Laplace effect with

$$\frac{M_E}{M_E^{in}} - \frac{M_M}{M_M^{in}} \approx 2a_2 \left(\frac{W_{grav}}{c^2 M}|_E - \frac{W_{grav}}{c^2 M}|_M\right)$$

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Lessons : In "modified" theories of gravity which are "minimally coupled"

• the WEP is satisfied for bodies with small gravitational energy

• the SEP is violated : the motion of, say, the Earth/Moon system is modified by the presence of the "spectator" Sun

• this induces a "Laplace effect"

II. BEYOND NEWTON: The example of Nordström's theories

(or : gravity as a scalar field in **flat space-time**, as revisited by Einstein)

- "$E = mc^2$". Hence $m_H^{in} = m_p^{in} + m_e^{in} - 13.6$eV.

Matter described by $T_{\mu\nu} = (\epsilon^{in} + p)u_\mu u_\nu + p\eta_{\mu\nu}$, $\epsilon^{in} \equiv \Sigma_a m_a^{in} c^2 / V$

- Gravity is described by a scalar potential $\Phi(x^\mu)$.

Outside a star : $\Phi = \Phi(x^\mu, M)$ where $M = M(\epsilon^{in}_{central}, EOS, R)$

- Define M^{in} as : $M^{in}c^2 = \int_{star} dV \epsilon^{in} + \int_{all\ space} dV\, T_g{}^{00}$

Question : $M = M^{in}$?

Answer depends on the theory

(Yes for Nordström's "linear" theory)

Nordström's theories, 1914

$$S_g = -\frac{1}{\kappa}\int d^4x\,\sqrt{-\ell}\,\ell^{\mu\nu}\partial_\mu\Phi\,\partial_\nu\Phi,$$ where $\ell_{\mu\nu}$ is Minkowski's metric

$$S_m = \int d^4x\,\mathcal{L}_m, \quad \mathcal{L}_m = \mathcal{L}_m[\psi_m, \tilde{\ell}_{\mu\nu}]$$ with $\tilde{\ell}_{\mu\nu} = (1+F(\Phi))^2\ell_{\mu\nu}$

Example: $S_m = -\Sigma_a m_a^{in} c^2 \int (1+F(\Phi))d\tau$

where m^{in} is the inertial mass of elementary objects (say H-atoms, including electromagnetic binding energy).

Hence a "minimally" coupled theory ($m^{in} = m$)
with "non-linear" coupling : $F(\Phi) = \Phi + 2a_2\Phi^2 + \cdots$)

field EOM : $D^2\Phi = -\frac{\kappa F'}{2(1+F)}T$ (replaces : $\triangle U = 4\pi G\rho^{in}\frac{dV}{dU}$)

$$D_\nu T^{\mu\nu} = \frac{F'}{1+F}T\,\partial^\mu\Phi \quad \text{or} \quad \frac{Du^\mu}{d\tau} = -\frac{F'}{1+F}(\partial^\mu\Phi + u^\mu u^\nu\partial_\nu\Phi)$$

(replaces : Euler equation : $\nabla p = -\rho^{in}\nabla U$ or $\ddot{\vec{r}} = -\nabla U$.)

Define the inertial mass of a gravitationally bound system

(not as straightforward as in Newton's physics where $M^{in} = \int dV \rho^{in}$)

It follows form EOM (and action) that

$$D_\nu (T_g^{\mu\nu} + T_m^{\mu\nu}) = 0 \quad \text{with } T^g_{\mu\nu} = \frac{2}{\kappa}\left(\partial_\mu \Phi \, \partial_\nu \Phi - \frac{1}{2}\ell_{\mu\nu} \, \partial^\rho \Phi \partial_\rho \Phi\right).$$

Integrating over space in an *inertial frame* ($\ell_{\mu\nu} = \eta_{\mu\nu}$) one has

$$\frac{dM^{in}}{dt} = \int_{S_\infty} T^g_{0i} n^i dS \quad \text{with} \quad M^{in} \equiv \int dV (T_g^{00} + T_m^{00}).$$

M^{in} is constant if the system does not radiates gravitational waves

It is a "conserved charge"

In the inertial system where the 3-momentum is zero

M^{in} is the inertial mass of the system

(it includes the energy of the gravitational field)

question: do we have $M^{in} = M$? If not what are the EOM of 'big' bodies?

- “Inner problem” : *the field created by an extended object*

Specialize the eom and definition of M^{in} to a static, perfect fluid, spherically symmetric configuration :

Choose an equation of state (e.g. $\epsilon^{in} = cst.$)

Expand: $F(\Phi) = \Phi + \frac{1}{2}a_2\Phi^2 + \cdots$. Solve by iteration. Find, for $r \geq R$:

$$\Phi = -\frac{GM}{c^2 r} \quad \text{with} \quad M = M^{in}\left(1 + 2a_2 \frac{W^{grav}_{Newton}}{Mc^2}\right)$$

$$\text{where } W^{grav}_{Newton} \equiv \frac{1}{2}\int \epsilon^{in}\Phi\, dV \left(= -\frac{3GM^2}{5R}\right)$$

in nice agreement with the result in Newton’s modified gravity.

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One can show that $M = M^{in}$ if $F(\Phi) = \Phi$ at all orders.

Hence the claim that the linearly coupled Nordström theory obeys the SEP.

• Second step (“N-body pb”) : *“skeletonization” of N extended bodies*

If one can ignore W^{grav}_{Newton}. Extended objects behave as “probes” :

$$\frac{Du^\mu}{d\tau} = -\frac{dF/d\Phi}{1+F(\Phi)} \left(c^2 \partial^\mu \Phi + u^\mu u^\nu \partial_\nu \Phi \right) \quad \text{(WEP is satisfied)}$$

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Integrate, with one-body $\Phi = -\frac{GM_\odot}{c^2 r}$, $\quad F(\Phi) = \Phi + \frac{1}{2} a_2 \Phi^2 + \cdots$

Compute bending of light and perihelion shift. Find PPN parameters :

$\gamma = -1$, $\beta = \frac{1+a_2}{2}$ (hence Nordström’s theory is experimentally ruled out)

and also : $2a_2 = 4\beta - \gamma - 3 \equiv \eta$

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modification of Kepler’s third law : $\frac{4\pi^2 a_a^3}{P_a^2} \approx GM_\odot \left(1 - a_2 \frac{GM_a}{c^2 r_a}\right)$
as in Newton’s modified gravity (P_a measured with inertial frame time).

The difference M/M^{in} cannot be ignored:

$$M = M^{in}(1 + 2a_2\epsilon) \quad \text{with } \epsilon = \frac{W_{grav}^{Newton}}{c^2 M} \text{ constant}$$

At lowest order the computation is the same as in modified Newton's theory and corresponding effects add up linearly. Thus :

modification of Kepler's third law : $\frac{4\pi^2 a_a^3}{P_a^2} \approx GM_\odot \left[1 + 2a_2 \left(\epsilon_a - \frac{GM_a}{2c^2 r_a}\right)\right]$

Extra-terms are negligible. Hence a "Laplace-Nordtvedt" effect with

$$\frac{M_E}{M_E^{in}} - \frac{M_M}{M_M^{in}} = \eta \left(\frac{W_{grav}^{Newton}}{c^2 M}\Big|_E - \frac{W_{grav}^{Newton}}{c^2 M}\Big|_M + \frac{G(M_E - M_M)}{2c^2 r_{EM}} \right)$$

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Lessons : In Nordström's, just like in Newtonian "modified" theories :

- the WEP is satisfied for bodies with small gravitational energy
- the SEP is violated UNLESS matter is linearly coupled to gravity.

Nordström's theory : "Jordan frame" formulation

Recall: $S_g = -\frac{1}{\kappa}\int d^4x\,\sqrt{-\ell}\,\ell^{\mu\nu}\partial_\mu\Phi\,\partial_\nu\Phi$, with $\ell_{\mu\nu}$ Minkowski's metric

$S_m = \int d^4x\,\mathcal{L}_m$, $\quad \mathcal{L}_m = \mathcal{L}_m[\psi_m, \tilde{\ell}_{\mu\nu}]$ with $\tilde{\ell}_{\mu\nu} = (1+F(\Phi))^2\ell_{\mu\nu}$

EOM can be rewritten in terms of $\tilde{\ell}_{\mu\nu}$ as (Einstein-Fokker 1914) :

- $\tilde{R} = 3\kappa\left(\frac{dF}{d\Phi}\right)^2\tilde{T}_m - \frac{6}{1+F}\frac{d^2F}{d\Phi^2}\,\tilde{\ell}^{\mu\nu}\partial_\mu\Phi\,\partial_\nu\Phi$

  (hence : $R = 3\kappa\tilde{T}_m$ if $F(\Phi) = \Phi$, that is, when SEP is satisfied)

- $\tilde{D}_\nu\tilde{T}_m^{\mu\nu} = 0$ with $T_{\mu\nu}^{fluid} = (\epsilon^{in} + p)\tilde{u}_\mu\tilde{u}_\nu/c^2 + p\,\tilde{\ell}_{\mu\nu}$ or $\frac{\tilde{D}\tilde{u}^\mu}{d\tilde{\tau}} = 0$

  (WEP is satisfied for bodies with negligible gravitational binding energy)

- Steps one-two-three :

Static, perfect fluid, spherically symmetric configuration : same EOM as in the original, special relativistic, Nordström's formulation.

However, a definition of the inertial mass is *not* straightforward since a Jordan frame lagrangian is required.

In case of linear coupling ($F(\Phi) = \Phi$) this is possible (ND 2011). The inertial mass is given by an integral on the 2-sphere at infinity and $M = M^{in}$.

In this case then : no deviation from Kepler's 3rd law, no Nordtvedt effect.

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Lesson :
In order to compute possible observable violations of the SEP in relativistic theories of gravity, it is essential to be able to define the inertial mass of an extended self-gravitating object.

III. Beyond Nordström : SCALAR-TENSOR THEORIES

(in the "simple" case of Brans-Dicke theory)

- Jordan frame (that is, the "Einstein-Fokker" formulation) :

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(\phi R - \frac{\omega}{\phi}\partial^\mu\phi\partial_\mu\phi\right) + S_m[\psi_m\ ;\ g_{\mu\nu}]$$

$$G_{\mu\nu} = \frac{8\pi}{\phi}T_{\mu\nu} + \frac{\omega}{\phi^2}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\partial^\rho\phi\partial_\rho\phi\right) + \frac{1}{\phi}(D_{\mu\nu}\phi - g_{\mu\nu}D^\rho D_\rho\phi)$$

$$D^\rho D_\rho\phi = \frac{8\pi}{3+2\omega}T \quad \text{with} \quad T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}.$$

- Einstein frame (that is, the "Nordström" formulation) :

$$S = \frac{1}{16\pi G_*}\int d^4x\sqrt{-g_*}\,(R_* - 2\partial^\mu\psi\partial_\mu\psi) + S_m[\psi_m\ ;\ g_{\mu\nu}] + b.t.$$

$$R^*_{\mu\nu} = 2\partial_\mu\psi\partial_\nu\psi + T^*_{\mu\nu} - \frac{1}{2}g^*_{\mu\nu}T^* \quad , \quad D^\rho_* D^*_\rho\psi = \frac{4\pi G_*}{\sqrt{3+2\omega}}T^*$$

- From Jordan to Einstein :

$$g^*_{\mu\nu} = (G_*\phi)\,g_{\mu\nu}\ ,\ \psi = \frac{\sqrt{3+2\omega}}{2}\ln(G_*\phi) \quad (\text{with}\ \ G^*\phi_\infty = 1)$$

To cut a long story short:

- Find the static, spherically symmetric, vacuum solution in Jordan frame :

$$\to -\left(1 - \frac{2M}{r} + \frac{2M^2}{r^2} + \cdots\right) dt^2 + \left(1 + \frac{2(M-2M_s)}{r} + \cdots\right) d\vec{x}^2$$

M_s depends on the inner structure ($M_s = 0$ for a BH)

Test bodies with negligible gravitational energy follow geodesics

Hence M is the gravitational mass of the body

- Define the inertial mass of an extended body

  Via various "pseudo-tensors" constructed in the Jordan frame

  Via rigorously defined charges (ADM etc) in the Einstein frame

  Via rigorously defined charges (e.g. Katz') in the Jordan frame

All give $M^{in} = M - M_s$

• "Skeletonize" N extended bodies (Eardley 1975, Zaglauer 1992)

A gravitationally bound body "swells and shrinks" with variations in the exterior potential so that its inertial mass M_a^{in} varies.

Describe it as a point particle with action $S_m = -\Sigma_a \int M_a^{in}(\phi) d\tau$

with $M_a^{in} = M_a^{\infty}\left(1 + s_a \frac{\phi-\phi_\infty}{\phi_\infty} + \cdots\right)$ where $s = \frac{\partial \ln M^{in}}{\partial \ln \phi}|_N$

is the "sensitivity" of the body. (For a BH: $s = 1/2$.)

• Solve the EOM and find a Laplace Nordtvedt effect:
(see e.g. Berti et al. Appendix A, 1112.4903)

$\ddot{\vec{r}}_{ME} = -\frac{M_* \vec{r}_{ME}}{r^3_{ME}} + \eta(1-2s_S)(s_E - s_M)\frac{M_S \vec{r}_{ES}}{r^3_{ES}}$ where $\eta = \frac{1}{2+\omega}$

If, $s \approx \frac{W^{grav}_{Newton}}{M}$, then $s_S \ll 1$ and the effect is the same as the one obtained in Nordström's gravity.
If S is a BH, no Nordtvedt effect. BH follow geodesics of the Einstein frame, not Jordan's.

SUMMARY and CONCLUSIONS

1. Modified gravity theories generically predict violation of the SEP :
for gravitationally bound systems, $M^{in} \neq M$

• where M is read off the asymptotic form of g_{00} in the "Jordan frame" where "ordinary" test objects follow geodesics.

• where M^{in} has to be defined as a "global Noether charge"

2. The problem of motion has to be revisited (since $M^{in} \neq M$)

• Gravitationally bound objects are described as point particles with an inertial M^{in}, which depends on the value of the scalar field.
Hence they no longer follow geodesics.

• Observational consequences (such as the Nordtvedt effect and dipole gravitational radiation) have been and still are studied.

The lesson :

Conventional wisdom: $E = mc^2$ with $m = m^{in}$. *All forms of energy gravitate*, including vacuum energy. Hence : $G_{\mu\nu} = \kappa(T_{\mu\nu} + \Lambda^{vac} g_{\mu\nu})$ with "catastrophic" Λ^{vac}.

However : "All forms of energy gravitate" IF $m^{in} = m$, which is generally NOT true in theories other than Newton's, Nordström's or Einstein's.

Hence Vacuum (inertial) energy may gravitate in a non trivial way.

Thank you for your attention